Comment on "Localized vortices with a semi-integer charge in nonlinear dynamical lattices"

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(Received 22 December 2001; published 24 October 2002)

In a recent paper by Kevrekidis, Malomed, Bishop, and Frantzeskakis [Phys. Rev. E **65**, 016605 (2001)] the existence of localized vortices with semi-integer topological charge as exact stationary solutions in a two-dimensional discrete nonlinear Schrödinger model is claimed, as well as the existence of an analog solution in the one-dimensional model. We point out that the existence of such exact stationary solutions would violate fundamental conservation laws, and therefore these claims are erroneous and appear as a consequence of inaccurate numerics. We illustrate the origin of these errors by performing similar numerical calculations using more accurate numerics.

DOI: 10.1103/PhysRevE.66.048601 PACS number(s): 41.20.Jb, 63.20.Pw

After it was originally suggested by Aubry and co-worker [1,2], it is by now a well-known fact that two-dimensional (2D) nonlinear Hamiltonian lattices may sustain exact, linearly stable solutions of the vortex type, which are spatially exponentially localized and time periodic, and carry an energy current by means of a phase torsion in a closed loop. Such solutions were explicitly calculated numerically to high precision, using Newton-type schemes, for Klein-Gordon [3] as well as discrete nonlinear Schrödinger (DNLS) [4] lattices, and their linear stability for small intersite coupling was also explicitly demonstrated. Some recent publications [5] have revisited this topic and provided some additional explicit examples of discrete localized vortices in DNLS lattices and illustrated explicitly the mechanisms of the instability that, as was mentioned already in [3,4], may occur when the intersite coupling is increased.

However, in the very recent Ref. [6], to which this comment pertains, Kevrekidis, Malomed, Bishop, and Frantzeskakis claim to demonstrate a very surprising result which, if true, would go beyond all examples of discrete localized vortices found earlier: the existence of an exact stable stationary vortex with semi-integer topological charge, $S=1/2$, in a DNLS lattice. Unfortunately, the claim is false, and, as we point out below, contradicts fundamental conservation laws of the DNLS model. The erroneous claim is based on using numerical schemes which, as we illustrate below, are not accurate enough to determine whether the particular configurations obtained in Ref. [6], for some rather extreme parameter values close to the uncoupled limit, are exact or only approximate solutions to the DNLS equation.

The explicit example proposed in Ref. [6], described by Eqs. (9) and Fig. 1 in Ref. [6], is quasi-one-dimensional (quasi-1D) and consists, in the "anticontinuous" limit $C=0$ [using notations as in Eqs. (1)–(6) in Ref. [6]], of two sites with equal nonzero amplitude, phase shifted by $\pi/2$, situated along a row in the 2D DNLS lattice, while all other sites at $C=0$ have amplitude zero. Since the configuration is quasi-1D, the authors of Ref. [6] also consider in Sec. III of their paper its analog in a 1D lattice. We should note that in all the explicit examples given in Ref. [6], the distance between the sites of nonzero amplitude is rather large, apparently six sites in Fig. 3 and ten sites in Fig. 5 (although the text claims a distance of six sites also in the latter figure).

As the configuration is quasi-1D, we will here for simplicity first formulate our criticisms in the framework of the 1D model, and then show how the same arguments can be extended also to the 2D case. The 1D DNLS equation [Eq. (1) in Ref. [6]] has two conserved quantities, norm $\mathcal{N}$ and Hamiltonian H, with corresponding flux densities $J_{\mathcal{N}}$ and J_H (see, e.g., Ref. [7]):

$$\mathcal{N}=\sum_n |\psi_n|^2, \quad J_{\mathcal{N}}=2C\,\mathrm{Im}[\psi_n^*\psi_{n+1}], \tag{1}$$

$$H=\sum_n \left(C|\psi_{n+1}-\psi_n|^2-\frac{1}{2}|\psi_n|^4\right),$$

$$J_H=-2C\,\mathrm{Re}[\dot{\psi}_{n+1}(\psi_{n+1}^*-\psi_n^*)]. \tag{2}$$

For a stationary, time-periodic solution of the form $\psi_n=\exp(i\Lambda t)u_n$ as considered in Ref. [6] [Eq. (5)], the flux (current) densities take the form $J_{\mathcal{N}}=2C\,\mathrm{Im}(u_{n+1}u_n^*)$ and $J_H=-\Lambda J_{\mathcal{N}}$. As for such solutions (which are not the only kinds of solutions that might yield localized vortices in 2D [4] but the only kind considered in Ref. [6]) the norm density $|\psi_n|^2$ is constant in time for each site n, the net flow of incoming and outgoing current density $J_{\mathcal{N}}$ from sites $n\pm 1$ must be zero at each site n (the corresponding is of course also true in 2D taking into account the four possible directions). Now, separating amplitudes and phases by writing $u_n=|u_n|\exp(i\phi_n)$ yields $J_{\mathcal{N}}=2C|u_{n+1}||u_n|\sin(\phi_{n+1}-\phi_n)$, so that, just as in standard quantum mechanics, any nontrivial phase gradient creates a current. Thus, as for a stationary state, $J_{\mathcal{N}}$ must be constant everywhere, we can directly conclude that for any localized solution for which $|u_n|\to 0$, $|n|\to\infty$ we must have $J_{\mathcal{N}}\equiv 0$, and thus the only possible phase gradients when $C\neq 0$ are $\phi_{n+1}-\phi_n=0,\pi$ (the particular case $u_{2k}\equiv 0$ for which $J_{\mathcal{N}}\equiv 0$ must necessarily be periodic and therefore nonlocalized). Thus, we conclude that the solutions proposed in Ref. [6], Sec. III, having a total phase gradient of $\pi/2$ over six (or ten) sites distance, cannot be

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exact stationary solutions to the 1D DNLS equation as their existence would violate the conservation laws (1), (2).

The same argument carries directly over to quasi-1D configurations in the 2D DNLS equation, since no net current can flow in the perpendicular directions for symmetry reasons. Consequently, also the vortices with semi-integer topological charge proposed in Sec. II of Ref. [6] cannot exist as exact solutions. To make this argument explicit, consider a hypothetical stationary solution of the form proposed in Eq. (8) of Ref. [6]: $u_{m,n}=|u_{m,n}|\exp(i\phi_{m,n})$ with $\phi_{m,n}\sim-|\theta|/2$, where θ is the polar angle in the lattice plane, measured so that $\theta=0$ defines the line between the two sites with nonzero amplitude for $C=0$ ("row $m=10$" in the notation of Ref. [6]), and the origin is halfway between these sites. Consider then the total current flow through an infinite "vertical" line, chosen, e.g., as the "column $n=10$" in the notation of Ref. [6] and defined by $\theta=\pm\pi/2$. Because of its very construction, the solution is symmetric with respect to reflection in the "horizontal" line "$m=10$" (cf. Figs. 1 and 3 in Ref. [6]), and therefore the contributions to the total current from rows in the "upper" half plane ($m>10$) must be identical to, and *in the same direction as*, the contributions from the corresponding rows in the "lower" half plane ($m<10$). Since for each row m such a hypothetical solution must have a nontrivial phase gradient between any sites n and $n+1$ ($0<\phi_{m,n+1}-\phi_{m,n}<\pi/2$ by construction) yielding a nonzero horizontal current just as in the 1D case, with the same (positive) sign for all rows, the total current summed over all rows across the vertical line must also be nonzero. But since the solution is assumed to be localized, by the same argument as in the 1D case there can be no current at infinity, and thus such a solution cannot be stationary since the norm (and Hamiltonian) current flowing from one half plane to the other is uncompensated. More generally, in other models of anharmonic Hamiltonian lattices where energy is the only conserved quantity, the same kind of argument applies relating phase gradients of time-periodic solutions to energy currents [2,3].

Although the simple argument above immediately disproves the claims of existence of localized quasi-1D modes with a phase gradient in Ref. [6], it is still instructive to trace the origin of these erroneous claims. We first note that the tail of a single one-site breather in the 1D DNLS equation decays exponentially as $u_n\sim\exp(-\beta|n|)$, where $\cosh\beta=\Lambda/(2C)+1$. Using, for example, the parameter values indicated in Fig. 5 of Ref. [6] ($C=0.005$ and $\Lambda=0.32$), we obtain $\beta\approx4.2$ so that over a distance of six sites the breather would decay with a factor $\sim10^{-11}$ and over ten sites with $\sim10^{-18}$. Thus, the interaction between two breathers inserted at such distances as in Ref. [6] would be extremely weak, and, with the numerical accuracy reported in the paper (10^{-8}), an approximate solution with an arbitrary relative phase between the two breathers could easily be mistaken for an exact stationary solution, since the current $J_{\mathcal{N}}$ that would flow would be very small but always nonzero, unless the relative phases were 0 or π. A clear sign of the insufficient numerical accuracy (for the 2D case) can be seen in Fig. 2 of Ref. [6], where the stability analysis shows *two* pairs of stability eigenvalues which are closer to zero than the claimed

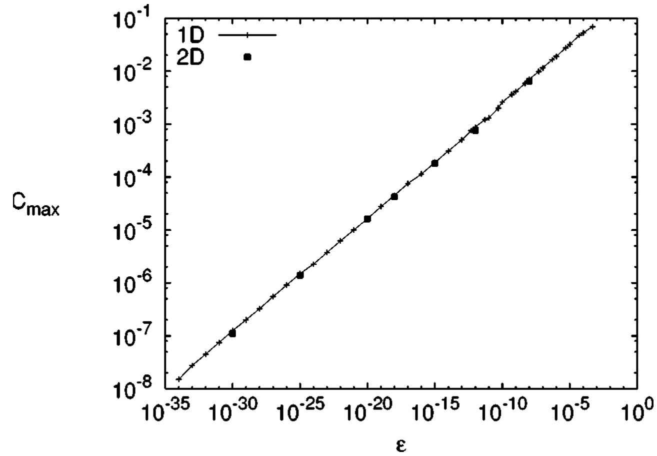

FIG. 1. Line with crosses: Maximum value of C for which the Newton algorithm "converges" to a solution that is an "exact" stationary solution to the 1D DNLS equation to accuracy ϵ with a $\pi/2$ phase shift between two breathers placed six sites apart ($\Lambda=0.32$ as in Ref. [6]). The convergence criterion used is $\Sigma_n[|(\Lambda-|u_n|^2)x_n-C\Delta_2x_n|+|(\Lambda-|u_n|^2)y_n-C\Delta_2y_n|]<\epsilon$, where $u_n=x_n+iy_n$; x_n,y_n real. The approximate relation is $C_{max}\sim\epsilon^{0.215}$. Filled squares: Corresponding results for the 2D case (only a few points are calculated since the 2D calculations in quadruple precision are computationally expensive).

numerical accuracy of the solution 10^{-8}. In the absence of additional symmetries, a stationary solution normally only has one zero eigenvalue corresponding to the overall phase degeneracy; the existence of a second zero (to the used precision) eigenvalue here, not related to any symmetries, indicates that the Newton scheme has not yet converged to an exact solution.

To ultimately resolve the apparent conflict between the numerical claims in Ref. [6] and the general nonexistence of localized 1D or quasi-1D stationary solutions with a phase gradient, we have attempted to calculate the same kind of solutions as reported in Sec. III of Ref. [6], for the same parameters, using the same kind of Newton method as described in Ref. [6] but with higher accuracy, using when necessary quadruple precision arithmetics. We have found that the maximum value of C for which the numerical scheme converges, within the prescribed accuracy, to a solution with a $\pi/2$ phase difference between the constituting breathers as expected decreases towards zero when the allowed error of the solution is decreased. Typical results are illustrated in Fig. 1. We have also, in the same way, attempted to calculate the vortices with semi-integer topological charge reported in Sec. II B of Ref. [6] for the 2D model, with very similar results as for the 1D case (filled squares in Fig. 1). (As described in Ref. [3] other numerical schemes using, e.g., singular value decomposition would, in principle, be more suitable here and have also been implemented with similar results; however we refrain from discussing this here to facilitate comparison with the results of Ref. [6].)

I thank S. Aubry and G. Kopidakis for many enlightening discussions during several years of collaboration, and acknowledge financial support from the Swedish Research Council.

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